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Invariant Measures and Decay of Correlations for a Class of Ergodic Probabilistic Cellular Automata

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Abstract We give new sufficient ergodicity conditions for two-state probabilistic cellular automata (PCA) of any dimension and any radius. The proof of this result is based on an extended version of the duality concept. Under these assumptions, in the one dimensional case, we study some properties of the unique invariant measure and show that it is shift-mixing. Also, the decay of correlation is studied in detail. In this sense, the extended concept of duality gives exponential decay of correlation and allows to compute explicitly all the constants involved.

Keywords Probabilistic cellular automata · Invariant measures · Duality decay of correlation

1 Introduction

Probabilistic cellular automata (PCA) are discrete time Markov processes which have been intensely studied since at least Stavskaja and Pjatetskii-Shapiro [12] (1968). This kind of processes have as state space a product space $X = A^{\mathbb{Z}^d}$ where A is any finite set and d is any positive integer. We may regard a PCA as an interacting particle system where particles update its states simultaneously and independently. Recall that a PCA is ergodic if there exists only one invariant measure μ and starting from any initial measure μ_0 the system converges to μ.

The aim of this paper is to use duality principles to study the ergodicity of two-state PCA. More precisely our work gives new sufficient ergodicity conditions for the expression of the PCA's local transition probabilities (see Theorem 2) and show that under these conditions the invariant measure is shift-mixing with exponential decay of correlation. Relations between

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the PCA and the dual process (see Lemmas 4 and 2) also allow us to give a very simple expression of the constant of the decay of correlation as a function of the radius (of the PCA) and the transition probabilities of the PCA (see Theorem 3). Moreover the proof of Theorem 2 shows in detail how to compute the value of the invariant measure on cylinders. Results about the decay of correlation is an answer to a question raised in [9].

The existence of a dual process satisfying the duality equation (see Definition 1 and Liggett [7]) gives useful information (problems in uncountable sets can be reformulated as problems in countable sets) about the PCA but is not always sufficient to prove that a PCA is ergodic. In [9], Lopez, Sanz and Sobottka introduced an extended concept of duality (see Definition 2) and gave general results about ergodicity (see Theorem 1). They used this powerful general theory to give results on multi-state one-dimensional PCA of radius one and extended previous results about the *Domany-Kinzel model* (see [1] for an introduction and [5] for extensions). Previously, in [6] Konno has given ergodicity conditions for multi-state one-dimensional PCA using self-duality equations.

Even if, in some cases, the existence of null transition probabilities allows to prove ergodicity of a certain class of PCA (see [5] and [6]), it had been conjectured that in the one-dimensional case positive noise cellular automata are ergodic. However, P. Gacks, in 2000, introduced a very complex counterexample (see [3] and [4]) for noisy deterministic cellular automata. In that case, the noisy one-dimensional cellular automata does not forget the past and starting from different initial distribution, the PCA may converge to different invariant measures. His result can be extended to noisy PCA with positive rates. This conjecture was formulated only in the one-dimensional case since in dimension 2 or higher, it is easier to show the existence of at least two invariant measures. For instance, the two-dimensional Ising model [4] or the Toom example (see [13]) that exhibit *eroder* properties. From Theorem 2 there exists a subclass of *attractive* PCA (class $\mathfrak{C}$) where the noisy conjecture is verified ($p(I_r) < 1$ implies ergodicity).

In [9], the authors describe some ergodicity conditions for multi-state PCA. When the number of states is greater than 2, the ergodicity conditions are rather restrictive in order to be able to give general results. More general ergodicity conditions are interesting (see [9], Sect. 3.2) but seems to be very complex when the radius of the PCA grows. In this paper, we restrict the study to the two state case, which allows to show more easily general results for PCA of any radius. These sufficient ergodicity conditions can be compared to the Shlosman-Dobrushin condition applied to PCA (see [10] and [11]). In some examples (see Sect. 3.1) our sufficient conditions induced by the concept of duality allow to show ergodicity and decay of correlations where the Dobrushin conditions can not be applied. Moreover, for some classes of ergodic PCA Theorem 3 gives greater constants for the decay of spatial correlation.

This paper is organized as follows. In Sect. 2, we present the basic definitions, notations and some preliminary results. In Sect. 3, we state the main results, Theorems 2 and 3. We prove Theorem 2 in Sect. 4. We conclude the paper in Sect. 5 with the proof of the decay of correlation.

2 Definitions, Notations and Preliminary Results

2.1 Probabilistic Cellular Automata

We give a brief description of the theory of PCA.

Let A be a nonempty finite set and $X = A^{\mathbb{Z}^d}$ be endowed with the product topology. A probabilistic cellular automata is a discrete time Markov process on the state space X. Let $M(X)$ be the set of probability measures on X.

Let R be a finite subset of $\mathbb{Z}^d$ of cardinality $|R|$ and f a map from $A^{|R|+1}$ to $[0, 1]$. The discrete time Markov process $\eta_{.} = \{\eta_t(z) \in A : t \in \mathbb{N}, z \in \mathbb{Z}^d\}$ whose evolution satisfies

$$\mathbb{P}\left[\eta_{t+1}(z) = a|\eta_t(z+i) = b_i,\ \forall i \in R\right] = f\left(a, (b_i)_{i\in R}\right), \tag{1.1}$$

for all $t \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ is a well defined (discrete time) stochastic process which from now on will be called d-dimensional PCA. Here, $\mathbb{P}$ stands for the probability measure on $A^{\mathbb{Z}^d}$ induced by the family of local transition probabilities. Also, let $\mathbb{E}$ be the expectation operator with respect to this probability measure. For more details on the definition of PCA see Toom et al. [14], Maes and Shlosman [10] and Lopez and Sanz [8].

Let μ_0 be the initial distribution of the PCA. For any $t \geq 0$, we call μ_t the distribution of the process at time t. The measure μ_t is defined on cylinder $U = N(\Lambda, \phi) = \{\xi \in A^{\mathbb{Z}^d} : \xi(x) = \phi(x)\ \forall x \in \Lambda\}$ for some fixed $\phi \in A^{\mathbb{Z}^d}$ and $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ by

$$\mu_t(U) = \sum_{V\in\mathcal{C}_t} \mu_0(V)\mathbb{P}_{\eta_0\in V}\{\eta_t \in U\},$$

where $\mathcal{C}_t$ is the family of all cylinders of X on the coordinates of Λ (the finite subset of $\mathbb{Z}^d$ used to defined u).

In this paper the notation $|\Lambda|$ will represent the cardinality of Λ when Λ is a finite subset of $\mathbb{Z}^d$. If $U = N(\Xi, \phi)$ is a cylinder set, the notation $|U|$ will represent the cardinality $|\Xi|$ of the set $\Xi \subset \mathbb{Z}^d$. In the one dimensional case we adopt the following notation: For any sequence of letters $U = (u_0, \ldots, u_n) \in A^{n+1}$, the set $[U]_s = [u_0 \ldots u_n]_s := \{x \in A^{\mathbb{Z}}|x(s) = u_0, \ldots, x(s+n) = u_n\}$ will be called cylinder and $|U| = n+1$.

2.2 Two-state Probabilistic Cellular Automata

In order to simplify the notation we will focus our attention on two-state PCA, that is to say PCA $\eta_{.}$ on $\{0, 1\}^{\mathbb{Z}^d}$. For any finite set Y, denote by $\wp(Y)$ the set of all subsets of Y and for any positive integer r let us define

$$I_r := \{i = (i_1, \ldots, i_d) \in \mathbb{Z}^d : -r \leq i_1, \ldots, i_d \leq r\}.$$

Since there is only two states we can rewrite equation (1.1) as the family of transition probabilities $\{p(J) : J \in \wp(I_r)\}$ such that

$$p(J) := \mathbb{P}\{\eta_{t+1}(z) = 1|\eta_t(z+j) = 1 : j \in J\}.$$

Note that any PCA with state space $\{0, 1\}^{\mathbb{Z}^d}$ is completely characterized by a positive integer number r called the radius of the PCA and the set of transition probabilities $\{p(J) : J \in \wp(I_r)\}$.

2.3 The Invariant Probability Measure

Definition 1 Let T be a measure-preserving transformation of a probability space $(X, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ-algebra generated by the cylinder sets on X. We say that the probability measure μ is T-mixing if $\forall U, V \in \mathcal{F}$

$$\lim_{n\to\infty} \mu(U \cap T^{-n}V) = \mu(U)\mu(V).$$

Since the cylinder sets generate the σ-algebra $\mathcal{F}$, it follows that the measure μ is T-mixing when the last relation is satisfied by any pair of cylinder sets U and V (for more details see [15]).

2.4 Duality

The concept of duality is a powerful tool in the theory of interacting particle system. It provides relevant information about the evolution of the process under consideration from the study of other simpler process, the dual process. The reformulated problems may be more tractable than the original problems and some progress may be achieved. Now we give the (classical) definition of duality taken from [7].

Definition 2 Let $\eta_.$ and $\zeta_.$ be two Markov processes with state spaces X and Y respectively, and let $H(\eta, \zeta)$ be a bounded measurable function on $X \times Y$. The processes $\eta_.$ and $\zeta_.$ are said to be dual to one another with respect to H if

$$\mathbb{E}^{\eta}\left[H(\eta_t, \zeta)\right] = \mathbb{E}^{\zeta}\left[H(\eta, \zeta_t)\right]$$

for all $\eta \in X$ and $\zeta \in Y$.

Unfortunately, it is not true that every process has a dual. Recently, Lopez et al. [9] gave a new notion of duality which extends the previous one. More precisely, they gave the following definition.

Definition 3 Given two discrete time Markov processes, η_t with state space X and ζ_t with state space Y and $H : X \times Y \to \mathbb{R}$ and $\mathcal{D} : Y \to [0, \infty)$ bounded measurable functions, the process $\eta_.$ and $\zeta_.$ are said dual to one another with respect to $(H, \mathcal{D})$ if

$$\mathbb{E}_{\eta_0 = x}\left[H(\eta_1, y)\right] = \mathcal{D}(y)\mathbb{E}_{\zeta_0 = y}\left[H(x, \zeta_1)\right]. \tag{1.2}$$

2.5 Duality and Sufficient Ergodicity Conditions

In order to state our results in Sect. 3, we need to give the spirit and some elements of the proof of the following Theorem, which appears in [9].

Theorem 1 [9] *Suppose $\eta_.$ is a Markov process with state space $\mathbf{X}$ and $\xi_.$ is a Markov chain with countable state space $\mathbf{Y}$, which are dual to one another with respect to $(H, \mathcal{D})$. If $0 \le \mathcal{D}(y) < 1$ for all $y \in \mathbf{Y}$, then there exists a stochastic process $\tilde{\xi}_.$ with state space $\tilde{Y} = \mathbf{Y} \cup \{\mathcal{S}\}$ with $\mathcal{S}$ an extra state and a bounded measurable function $\tilde{H} : X \times \tilde{Y} \to \mathbb{R}$ such that $\eta_.$ and $\tilde{\xi}_.$ are dual to one another with respect to $\tilde{H}$. Furthermore, denoting by Θ the set of all absorbing states of $\xi_.$, if*

(i) *the set of linear combinations of $\{H(., y) : y \in \mathbf{Y}\}$ is dense in $C(X)$, the set of continuous maps from $\mathbf{X}$ to $\mathbb{R}$;*
(ii) *$\mathcal{D}(y) < 1$ for any $y \notin \Theta$, and $\mathbf{D} := \sup_{y \in \mathbf{Y} : \mathcal{D}(\mathbf{y}) < \mathbf{1}} \{\mathcal{D}(y)\} < 1$;*
(iii) *$H(., \theta) \equiv c(\theta)$ for all $\theta \in \Theta$ with $\mathcal{D}(\theta) = 1$;*

then $\eta_.$ is ergodic and its unique invariant measure is determined for any $y \in \mathbf{Y}$ by

$$\hat{\mu}(y) = \sum_{\theta \in \Theta,\, d(\theta) = 1} c(\theta)\mathbb{P}_{\tilde{\xi}_0 = y}\left[\tilde{\xi}_\tau = \theta\right], \tag{1.3}$$

where τ *is the hitting time of* $\tilde{\Theta} = \{\theta \in \Theta : \mathcal{D}(\theta) = 1\} \cup \{\mathcal{S}\}$ *for* $\tilde{\xi}_t$ *and* $\hat{\mu} = \lim_{t\to\infty} \hat{\mu}_t$ *with*

$$\hat{\mu}_t(y) = \int_{\mathbf{X}} H(x, y) d\mu_t(x).$$

Sketch of the Proof Suppose that there exists a dual process $\tilde{\xi}$ and a function $\tilde{H}$ that satisfies the following (classical) duality equation

$$\mathbb{E}_{\eta_0=x}\left[\tilde{H}(\eta_1, y)\right] = \mathbb{E}_{\tilde{\xi}_0=y}\left[\tilde{H}\left(x, \tilde{\xi}_1\right)\right]. \tag{1.4}$$

For all $s \in \mathbb{N}$ we write $\tilde{\mu}_s(y) := \int_{\mathbf{X}} \tilde{H}(x, y) d\mu_s(x)$ and it follows that

$$\begin{aligned}\tilde{\mu}_s(y) &= \int_{\mathbf{X}} \mathbb{E}_{\eta_0=x}[\tilde{H}(\eta_s, y)] d\mu(x) \overset{(1.3)}{=} \int_{\mathbf{X}} \mathbb{E}_{\xi_0=y}[\tilde{H}(x, \xi_s)] d\mu(x) \\ &= \mathbb{E}_{\xi_0=y}\left[\int_{\mathbf{X}} \tilde{H}(x, \xi_s) d\mu(x)\right] = \mathbb{E}_{\tilde{\xi}_0=y}[\tilde{\mu}(\tilde{\xi}_s)].\end{aligned}$$

Recall that τ is the hitting time of the dual process $\tilde{\xi}_\cdot$ entering an absorbing state $\theta \in \tilde{\Theta}$ and suppose that $\mathbb{P}\{\tau < \infty\} = 1$. It follows that

$$\begin{aligned}\lim_{s\to\infty} \tilde{\mu}_s(y) &= \lim_{s\to\infty} \sum_{\theta\in\tilde{\Theta}} \mathbb{E}_{\tilde{\xi}_0=y}[\tilde{\mu}(\tilde{\xi}_s) | \tilde{\xi}_t = \theta, \tau \le s] \mathbb{P}_{\tilde{\xi}_0=y}\{\tilde{\xi}_t = \theta, \tau \le s\} \\ &\quad + \lim_{s\to\infty} \mathbb{E}_{\tilde{\xi}_0=y}[\tilde{\mu}(\tilde{\xi}_s) | \tau > s] \mathbb{P}_{\tilde{\xi}_0=y}\{\tau > s\} \\ &= \sum_{\theta\in\tilde{\Theta}} \tilde{\mu}(\theta) \mathbb{P}_{\tilde{\xi}_0=y}\{\tilde{\xi}_\tau = \theta\}.\end{aligned}$$

Finally, when the set of linear combinations of the set $\{\tilde{H}(., y) | y \subset \mathbb{Z}^d\}$ is dense in $C(\mathbf{X})$ (the set of continuous functions from $\mathbf{X}$ to $\mathbb{R}$) the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges in the weak* topology. Also, the limit measure μ does not depend on the initial measure μ_0.

Hence, we have seen that the key point is to prove that $\mathbb{P}\{\tau < \infty\} = 1$. One way to show this, is to introduce the new type of duality (see (1.2)). If there exists a dual process $\xi_\cdot$ with respect to $(H, \mathcal{D})$ (see (1.2)), with state space $\mathbf{Y}$ and absorbing states space Θ that verifies the new concept of duality then we can define a standard dual process $\tilde{\xi}_\cdot$ with state space $\tilde{Y} = \mathbf{Y} \cup \{\mathcal{S}\}$ and such that the set of all absorbing states is $\tilde{\Theta} = \Theta \cup \{\mathcal{S}\}$. When the supremum on Y of the function $\mathcal{D}$ is less than one, we get that $\mathbb{P}\{\tau < \infty\} = 1$ (see Lemma 1). Here $\mathcal{S}$ is an extra absorbing state and the transition probabilities of $\tilde{\xi}_\cdot$ satisfy

$$\mathbb{P}_{\tilde{\xi}_0=\tilde{y}_0}\{\tilde{\xi}_1 = \tilde{y}_1\} = \begin{cases} \mathcal{D}(\tilde{y}_0) \mathbb{P}_{\xi_0=\tilde{y}_0}\{\xi_1 = \tilde{y}_1\}, & \text{if } \tilde{y}_0, \tilde{y}_1 \in \mathbf{Y} \\ 1 - \mathcal{D}(\tilde{y}_0), & \text{if } \tilde{y}_0 \in \mathbf{Y}, \tilde{y}_1 = \mathcal{S} \\ 1, & \text{if } \tilde{y}_0 = \tilde{y}_1 = \mathcal{S}. \end{cases}$$

Taking $\tilde{H}(x, y) = H(x, y)$ when $y \in \mathbf{Y}$ and $\tilde{H}(x, \mathcal{S}) = 0$ we get that $\tilde{\mu}(y) = \hat{\mu}(y)$ for all $y \in \mathbf{Y}$ and it is possible to show that the dual process $\tilde{\xi}_\cdot$ satisfies the standard duality equation (1.4). Note that since $\mathbf{D} = \sup_{y\in\mathbf{Y}:\mathcal{D}(y)<1}\{\mathcal{D}(y)\} < 1$, at each iteration the probability to enter the extra absorbing state $\mathcal{S}$ is positive and this implies the following result:

Lemma 1 *Under the conditions of Theorem* 1, *for all integer* $i \ge 1$ *one has*

$$\mathbb{P}(\tau > i) \le \mathbf{D}^i.$$

Proof By the Markov property we have that

$$\mathbb{P}_{\tilde{\xi_0}=\tilde{y_0}}\{\tau > i\} \le \mathbf{D} \times \mathbb{P}_{\tilde{\xi_0}=\tilde{y_0}}\{\tau > i-1\}.$$

Then, the result follows by using the mathematical induction principle. □

Note that Lemma 1 implies that $\mathbb{P}\{\tau < \infty\} = 1$ which finishes the proof of Theorem 1.

Before stating the main results of this paper, we introduce one more piece of notation: let ${}^{\infty}1^{\infty}$ denote the all one configuration, i.e. ${}^{\infty}1^{\infty} = (1_{\mathbb{Z}^d}(x))_{x\in\mathbb{Z}^d}$. Analogously, ${}^{\infty}0^{\infty}$ denote the all zero configuration.

3 Main Results and Examples

A PCA of radius r is called *attractive* if for any $J \subset I_r$ and $j \in I_r$ we have $p(J \cup \{j\}) \ge p(J)$. We consider here the following subclass of attractive PCA.

Definition 4 We say that a two-state PCA of radius r belongs to $\mathfrak{C}$ if its transition probabilities satisfy $p(J) = \sum_{J' \subseteq J} \lambda(J')$ for any $J \in \wp(I_r)$ where λ is some map from $\wp(I_r) \to [0, 1)$.

The following Proposition gives sufficient conditions for an attractive PCA to belong to $\mathfrak{C}$.

Proposition 1 *A two-state probabilistic cellular automaton $\eta_.$ belongs to $\mathfrak{C}$ if its transition probabilities satisfy the following set of inequalities*:

(a) *For any $i \in I_r$,*

$$p(\{i\}) \ge p(\emptyset).$$

(b) *For any $1 \le k \le |I_r| - 1$ and for any $j_0, \dots, j_k \in I_r$*

$$p(\{j_0, \dots, j_k\}) \ge (-1)^k p(\emptyset) - \sum_{n=0}^{k-1} (-1)^{k+1-n} \sum_{\{l_0,\dots,l_n\}\subset\{j_0,\dots,j_k\}} p(\{l_0, \dots l_n\}).$$

Theorem 2 *Let $\eta_.$ be a two-state d-dimensional probabilistic cellular automaton of radius r that belongs to $\mathfrak{C}$. If $p(I_r) < 1$ then $\eta_.$ is an ergodic PCA and there exists a dual process ξ which satisfy* (1.2). *Moreover, for any cylinder set U we can find $(\alpha_k \in \mathbb{Z})_{k\in K}$ and $(Y(k) \subset \mathbb{Z}^d)_{k\in K}$ with $|K| < \infty$ such that*

$$\mu(U) = \sum_{k\in K} \alpha_k \left(\sum_{l=1}^{\infty} \mathbb{P}_{\xi_0 = Y(k)}\{\xi_l = \emptyset | \xi_{l-1} \ne \emptyset\} \right).$$

Remark 1 In some cases it is possible to exchange the role of the two states $(0 \leftrightarrow 1)$ in order to show ergodicity using the previous results.

When $p(I_r) = 1$ the PCA may fail to be ergodic. For instance if $\lambda(\emptyset) = 0$ which means that the probability that a finite configuration of 0 gives 0 with probability one, there is at least two invariant measures: $\delta_{{}^{\infty}0^{\infty}}$ and $\delta_{{}^{\infty}1^{\infty}}$.

Corollary 1 *Under the conditions of Theorem 2 ($p(I_r) < 1$), if $\lambda(\emptyset) = 0$ then $\delta_{\infty 0^\infty}$ is the unique invariant measure.*

Theorem 3 *Let $\eta_.$ be a one-dimensional probabilistic cellular automaton $\in \mathfrak{C}$ of radius r with $p(I_r) =: \mathbf{D} \in [0, 1)$. Then, the unique invariant measure μ is shift-mixing. Also, if $\mathbf{D} \neq 0$, for any pair of cylinders $[U]_0 = [u_0 \ldots u_k]_0$, $[V]_0 = [v_0 \ldots v_{k'}]_0$ and $t \geq |U| + |V|$ we have*

$$|\mu([U]_0 \cap \sigma^{-t}[V]_0) - \mu([U]_0) \times \mu([V]_0)| \leq \exp(-a \times t) \times K(U, V),$$

where σ is the shift on $\{0, 1\}^{\mathbb{Z}}$, $a = 1/2r \times \ln(1/\mathbf{D})$ and $K(U, V)$ is a constant depending only on U, V, $\mathbf{D}$ and r.

Remark 2 This last result can be extended to d-dimensional PCA.

3.1 Examples and Comparison with Known Results

3.1.1 The Domany-Kinzel Model

This is a one-dimensional PCA $\eta_.$ of radius $r = 1$ introduced in [1] with transition probabilities

$$\begin{aligned}
\mathbb{P}\{\eta_{t+1}(z) = 1|\eta_t(z-1, z, z+1) = 000 \text{ or } 010\} &= p(\emptyset) = p(\{0\}) = a_0,\\
\mathbb{P}\{\eta_{t+1}(z) = 1|\eta_t(z-1, z, z+1) = 100 \text{ or } 110\} &= p(\{-1\})\\
&= p(\{-1, 0\}) = a_1,\\
\mathbb{P}\{\eta_{t+1}(z) = 1|\eta_t(z-1, z, z+1) = 001 \text{ or } 011\} &= p(\{1\}) = p(\{0, 1\}) = a_1
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}\{\eta_{t+1}(z) = 1|\eta_t(z-1, z, z+1) = 101 \text{ or } 111\} &= p(\{-1, 1\})\\
&= p(\{-1, 0, 1\}) = a_2,
\end{aligned}$$

where, for any subset $V \subset \mathbb{Z}$, $\eta(V) \in \{0, 1\}^V$ denote the restriction of a configuration $\eta \in \{0, 1\}^{\mathbb{Z}}$ to the set of positions in V.
Using Proposition 1 we obtain that $\eta_. \in \mathfrak{C}$ when $p(\{-1, 1\}) \geq p(\{-1\}) + p(\{1\}) - p(\emptyset)$, which is equivalent to the condition $a_2 \geq 2a_1 - a_0$. From Theorem 2 the PCA $\eta_.$ is ergodic if $p(I_r) = p(\{-1, 0, 1\}) = a_2 < 1$. From Theorem 3 the unique invariant measure is shift-mixing with exponential decay of spatial correlation such that for any pair of cylinders $[U]_0$ and $[V]_0$ and for all $t \geq |U| + |V|$ we obtain

$$|\mu([U]_0 \cap \sigma^{-t}[V]_0) - \mu([U]_0) \times \mu([V]_0)| \leq K \exp(-(1/2 \ln(1/a_2))t),$$

where K can be explicitly computed (see the end of Proof of Theorem 3). Using Theorem 2 we can compute, for instance, the measure of the cylinder $[01]_0$ which is

$$\begin{aligned}
\mu([01]_0) = \mu([1]_1) - \mu([11]_0) &= \hat{\mu}(\{1\}) - \hat{\mu}(\{0, 1\})\\
&= \sum_{t=1}^{\infty} \mathbb{P}_{\xi_0 = \{1\}}\{\xi_t = \emptyset | \xi_{t-1} \neq \emptyset\}
\end{aligned}$$

$$+\sum_{t=1}^{\infty}\mathbb{P}_{\xi_0=\{0,1\}}\{\xi_t=\emptyset|\xi_{t-1}\neq\emptyset\},$$

where $\xi_.$ is the associated dual process.

3.1.2 Two-dimensional Example

Let η be a two-state, two-dimensional PCA of radius one. In this case $I_1=\{(l,k)|-1\leq l,k\leq 1\}$ is a square of 9 sites. The transition probabilities $\{p(J)|J\subseteq I_1\}$ of $\eta_.$ are defined by $p(J)=\alpha\sum_{k=0}^{|J|}C_k^9=\alpha\times 2^{|J|}$ where C_k^l are the binomial coefficients. This PCA belongs to $\mathfrak{C}$ since for any $J\subseteq I_1$ we can write $\lambda(J)=\alpha$ and obtain that $P(J)=\sum_{J'\subseteq J}\lambda(J')$. This PCA is a kind of generalization to dimension 2 of the Domany-Kinzel model (each site has the same weight) with only one parameter. The sufficient ergodicity condition is $p(I_r)<1$ which implies that $\alpha\times 2^9<1$ ($\alpha<2^{-9}$) and the constant of decay of spatial correlation is $a=\frac{1}{2}\ln(1/(2^9\times\alpha))$.

3.1.3 Comparison with Dobrushin Condition

In [2], Dobrushin gives sufficient ergodicity conditions for interacting particle systems. Using our notation, these conditions applied to PCA can be translated as $\gamma<1$ (see [10] and [11]), where

$$\gamma=\sum_{j\in I_r}\sup_{J\subseteq I_r}|p(J\cup\{j\})-p(J)|.$$

In the case of the Domany-Kinsel model, which belongs to the class $\mathfrak{C}$, we obtain $\gamma=\sup_{J\subseteq I_r}|p(J\cup\{-1\})-p(J)|+\sup_{J\subseteq I_r}|p(J\cup\{1\})-p(J)|=2(a_2-a_1)$ since $\eta_.\in\mathfrak{C}$ ($a_2\geq 2a_1-a_0$). If $a_2<1$ (condition of Theorem 2) and $2(a_2-a_1)\geq 1$ the Dobrushin sufficient conditions can not be applied.

For the two-dimensional example we have $\gamma=\alpha(\sum_{k=1}^{9}k\times C_k^9)$. In this case $\gamma>p(I_r)$ and even if $\gamma<1$ the constant of decay of correlation $\frac{1}{2}\ln(1/(p(I_r))$ is greater than $\frac{1}{2}\ln(1/(\gamma))$, the constant of decay of correlation given in [10].

More generally, if a PCA belongs to $\mathfrak{C}$ the sufficient condition $p(I_r)<1$ can be rewritten as $p(I_r)=\sum_{J\subseteq I_r}\lambda(J)<1$ and the Dobrushin sufficient condition can be rewritten as $\gamma=\sum_{J\neq\emptyset,\,J\subseteq I_r}\lambda(J)\times|J|<1$.

4 Proof of Theorem 2 and Proposition 1

4.1 PCA's in $\mathfrak{C}$ and Their Dual Process

In [9], the authors give sufficient ergodicity conditions for one-dimensional multi-state PCA of radius one using a dual process satisfying (1.2). Here we will use an analogous dual process to give sufficient ergodicity conditions for two-state, d-dimensional PCA of radius r using the following duality equation:

$$\mathbb{E}_{\eta_0=x}[H(\eta_1,Y)]=\mathcal{D}(Y)\mathbb{E}_{\xi_0=Y}H[(x,\xi_1)],\tag{1.5}$$

where $\eta_.$ is a PCA with state space $\{0,1\}^{\mathbb{Z}^d}$. The state space of the dual process $\xi_.$ is the class of all finite subsets of $\mathbb{Z}^d$. As in [9] we define the function H by

$$H(x,Y)=\begin{cases}1, & \text{if } x(z)=1, \forall z\in Y\\ 0, & \text{otherwise.}\end{cases}$$

The rule for the evolution of the process ξ_t is given by

$$\xi_{t+1}=\bigcup_{z\in\xi_t}B(z)$$

where for any nonempty set $J\subseteq I_r$ we have

$$\mathbb{P}\Big[B(z)=\{z+j|j\in J\}\Big]=\pi(J)$$

and

$$\mathbb{P}\big[B(z)=\emptyset\big]=\pi(\emptyset).$$

Then, take the function $\mathcal{D}$ such that $\mathcal{D}(Y)=\mathbf{D}^{|Y|}$ for any finite subset $Y\subset\mathbb{Z}^d$, where $\mathbf{D}\in[0,1]$. Note that $\mathcal{D}(\emptyset)=1$ and $\emptyset$ is the unique absorbing state for this dual process.

4.2 The Functions H and $\hat{\mu}$

Note that, for this particular choice of H, we have

$$\hat{\mu}(\mathbb{Z}^d)=\int_X H(x,\mathbb{Z}^d)d\mu(x)=\mu(^{\infty}1^{\infty})=0$$

and

$$\hat{\mu}(\emptyset)=\int_X H(x,\emptyset)d\mu(x)=\mu(\{0,1\}^{\mathbb{Z}^d})=1,$$

where $X=\{0,1\}^{\mathbb{Z}^d}$ and $^{\infty}1^{\infty}$ is the all one configuration $(1_{\mathbb{Z}^d}(x))_{x\in\mathbb{Z}^d}$. The following Lemma is used in the proof of Theorems 2 and 3.

Lemma 2 *The set of linear combinations of* $\{H(.,y)|y\in\mathbb{Z}^d\}$ *is dense in* $C(\{0,1\}^{\mathbb{Z}^d},\mathbb{R})$, *the set of continuous function from* $\{0,1\}^{\mathbb{Z}^d}$ *to* $\mathbb{R}$. *For any cylinder* $U=N(\Lambda,\varphi)\subset\{0,1\}^{\mathbb{Z}^d}$ (*with* $\Lambda\subset\mathbb{Z}^d$, $|\Lambda|<\infty$ *and* $\varphi\in A^{\mathbb{Z}^d}$) *we have*

$$\mu(U)=\sum_{Y(i)}\alpha_i\hat{\mu}(Y(i)),$$

where $\alpha_i\in\mathbb{Z}$, $Y(i)\subset\mathbb{Z}^d$ *and* $\max\{|Y(i)|\}<\infty$.

Proof For the sake of simplicity, we only give the proof for the two-state, one-dimensional case. The key point of the proof consists in showing that any cylinder $[U]_t:=[u_0\ldots u_n]_t$, ($u_i\in\{0,1\}$ and $t,n\in\mathbb{N}$) can be decomposed into a non-commutative sequence of subtractions and unions of intersections of cylinders of the type $[1]_t$, $t\in\mathbb{Z}$. We denote by $T([U]_t)$ this decomposition. One way to accomplish this decomposition is to follow the following rules:

$$T([1]_t)=[1]_t,\qquad T([0]_t)=\{0,1\}^{\mathbb{Z}^d}-[1]_t.$$

Then, for all $t, n \in \mathbb{Z}$ and $U = u_0 \dots u_n$ we have

$$T([U1]_t) = T([U]_t) \cap [1]_{t+n+2}.$$

Thus,

$$T([U0]_t) = T([U]_t) - T([U]_t) \cap [1]_{t+2+n}.$$

For instance,

$$\begin{aligned} T([100]_0) &= T([10]_0) - T([101]_0) \\ &= (T([1]_0) - T([11]_0)) - (T([10]_0) \cap [1]_2) \\ &= [1]_0 - [11]_0 - (([1]_0 - [11]_0) \cap [1]_2) \\ &= ([1]_0 - [11]_0 - ([1]_0 \cap [1]_2)) \cup [111]_0. \end{aligned}$$

Then, note that $\mathbf{1}_{[1000]_0}$, the characteristic function of the cylinder $[1000]_0$, can be written as

$$\begin{aligned} \mathbf{1}_{[1000]_0}(x) &= \mathbf{1}_{[1]_0}(x) + \mathbf{1}_{[111]_0}(x) - \mathbf{1}_{[1]_0 \cap [1]_2}(x) - \mathbf{1}_{[11]_0}(x) \\ &= H(x, \{0\}) + H(x, \{0, 1, 2\}) - H(x, \{0, 2\}) - H(x, \{0, 1\}). \end{aligned}$$

Since for any finite subset $Y \subset \mathbb{Z}$ we have $\mathbf{1}_{\cap_{i \in Y}[1]_i}(x) = H(x, Y)$, it follows that for all $n \in \mathbb{N}$, $t \in \mathbb{Z}$ and $U \in \{0, 1\}^n$ we get $\mathbf{1}_{[U]_t} = \sum \alpha_i H(x, Y(i))$. This, in turn, implies that the set of linear combinations of the set $\{H(., Y) | Y \in \mathbb{Z}^d\}$ is dense in $C(\{0, 1\}^{\mathbb{Z}^d})$. We finish the proof by observing that for any cylinder $[U]_t$, we have

$$\begin{aligned} \mu([U]_t) &= \int \mathbf{1}_{[U]_t}(x) d\mu(x) \\ &= \int \sum \alpha_i H(x, Y(i)) d\mu(x) \\ &= \sum \alpha_i \hat{\mu}(Y(i)). \end{aligned}$$

□

Remark 3 Using the definition of H taken in [9] which takes into consideration the multi-state case, it is possible to prove Proposition 2 for more general d-dimensional PCA.

4.3 Proof of Theorem 2

We first establish a sequence of equalities between the transition probabilities of the PCA $(P(J) | J \in I_r)$ and the transition probabilities of the dual process $((\pi(J) | J \in I_r))$.

We can rewrite the right hand of (1.4) to obtain

$$\mathbb{E}_{\eta_0 = x}[H(\eta_1, Y)] = \mathbb{P}_{\eta_0 = x}\{\eta_1(z) = 1 \ \forall z \in Y\}.$$

Hence, using the independence property of $\eta_.$ we get that

$$\mathbb{P}_{\eta_0 = x}\{\eta_1(z) = 1 \ \forall z \in Y\} = \prod_{z \in Y} \mathbb{P}_{\eta_0 = x}\{\eta_1(z) = 1\}.$$

For the left hand of (1.5) we have

$$\mathbb{E}_{\xi_0 = Y}[H(x, \xi_1)] = \mathbb{P}_{\xi_0 = Y}\{x(z) = 1 \ , \forall z \in \xi_1\}.$$

For any $x \in \{0,1\}^{\mathbb{Z}^d}$ we denote by C_x the set $\{z \in \mathbb{Z}^d | x(z) = 1\}$. Then

$$\mathbb{P}_{\xi_0=Y}\{x(z) = 1\ \forall z \in \xi_1\} = \mathbb{P}_{\xi_0=Y}\{\xi_1 \subset C_x\}.$$

Using the independence property of the dual process we can assert that

$$\mathbb{P}_{\xi_0=Y}\{\xi_1 \subset C_x\} = \prod_{z\in Y} \mathbb{P}\{B(z) \subset C_x\}.$$

Finally we can rewrite (1.5) as

$$\begin{aligned}\prod_{z\in Y} \mathbb{P}_{\eta_0=x}\{\eta_1(z) = 1\} &= \mathbf{D}^{|Y|} \prod_{z\in Y} \mathbb{P}\{B(z) \subset C_x\}\\ &= \prod_{z\in Y} \mathbf{D} \times \mathbb{P}\{B(z) \subset C_x\} \qquad (1.6)\end{aligned}$$

which implies that

$$\prod_{z\in Y} p(J_z) = \prod_{z\in Y} \mathbf{D} \times \Delta_z, \qquad (1.7)$$

where $J_z = \{i - z | i \in \{C_x \cap \{j + z\} | j \in I_r\}\}$ and Δ_z is given by

$$\begin{aligned}\Delta_z = \pi(\emptyset) &+ \sum_{i\in I_r} \mathbf{1}_1(x(z+i)) \times \pi(\{i\})\\ &+ \sum_{i,j\in I_r} \mathbf{1}_{\{1\}}(x(z+i)) \times \mathbf{1}_{\{1\}}((x(z+j)) \times \pi(\{i,j\})\\ &+ \cdots + \sum_{i_1,\ldots,i_k\in I_r} \left(\prod_{l=1}^{k} \mathbf{1}_{\{1\}}(x(z+i_k))\right) \times \pi(\{i_1,\ldots,i_k\})\\ &+ \cdots + \left(\prod_{i\in I_r} \mathbf{1}_{\{1\}}(x(z+i))\right) \times \pi(I_r).\end{aligned}$$

By simplicity of notation we write $\pi(i_1,\ldots i_k)$ and $p(i_1,\ldots i_k)$ instead of $\pi(\{i_1,\ldots i_k\})$ and $p(\{i_1,\ldots i_k\})$.

Since (1.7) is true for all $x \in \{0,1\}^{\mathbb{Z}^d}$ we obtain the following equations for $\pi(.)$,

$$\begin{aligned}p(\emptyset) &= \mathbf{D}\pi(\emptyset)\\ p(i) &= \mathbf{D}[\pi(\emptyset) + \pi(i)]\\ p(i,j) &= \mathbf{D}[\pi(\emptyset) + \pi(i) + \pi(j) + \pi(i,j)]\\ p(i,j,k) &= \mathbf{D}[\pi(\emptyset) + \pi(i) + \pi(j) + \pi(i,j) + \pi(i,k) + \pi(j,k) + \pi(i,j,k)]\end{aligned}$$

where $i, j, k \in I_r$.

More generally, for any $0 \le k \le |I_r| - 1$,

$$p(i_0,\ldots,i_k) = \mathbf{D}\left[\pi(\emptyset) + \sum_{l=0}^{k} \pi(l) + \cdots + \sum_{i=0}^{k-1} \sum_{l_0,\ldots,l_i\in\{i_0,\ldots,i_k\}} \pi(l_0,\ldots,l_i) + \pi(l_0,l_1,\ldots,l_k)\right]. \qquad (1.8)$$

Since

$$\pi(\emptyset)+\sum_{k=0}^{|I_r|}\left(\sum_{l_0,l_1,\dots l_k\in I_r}\pi(l_0,l_1,\dots l_k)\right)+\pi(I_r)=1,$$

we get that $\mathbf{D}=p(I_r)$.

By definition, the dual process is completely determined by the parameters $0\le\pi(J)\le 1$ $(J\subseteq I_r)$. From the set of equations (1.8) the dual process associated with the particular functions H and $\mathcal{D}$ is well defined if the transition probabilities of the PCA satisfy $p(J)=\mathbf{D}\sum_{J\subseteq I_r}\pi(J)$ with $J\in\wp(I_r)$. In this case we get that $\lambda(J)=\mathbf{D}\pi(J)$ and we claim that a PCA $\eta_.$ admits a dual process that satisfies the duality equation (1.2) with particular functions H and $\mathcal{D}$ given in Sect. 4.1 if and only if this PCA belongs to the class $\mathfrak{C}$.

To show that the PCA is ergodic we need to verify the three conditions of Theorem 1. Condition (i) is verified since from Lemma 2, the set of linear combinations of functions belonging to $\{H(.,Y)|Y\in\mathbb{Z}^d\}$ is dense in $C(\{0,1\}^{\mathbb{Z}^d},\mathbb{R})$.
Condition (ii) is satisfied since $\sup_{Y\neq\emptyset}\{\mathcal{D}(Y)\}=\mathbf{D}=p(I_r)<1$.
Condition (iii) follows from the fact that $H(.,\emptyset)=1$ and $\mathcal{D}(\emptyset)=\mathbf{D}^{|\emptyset|}=1$.

Since $\emptyset$ is the only absorbing state for $\xi_.$, using Theorem 1 (1.2) we get that for any nonempty set $Y\subset\mathbb{Z}^d$

$$\hat{\mu}(Y)=\hat{\mu}(\emptyset)\mathbb{P}_{\xi_0=y}\{\xi_\tau=\emptyset\}=\sum_{t=1}^{+\infty}\mathbb{P}_{\xi_0=y}\{\xi_t=\emptyset|\xi_{t-1}\neq\emptyset\}.$$

From Lemma 2, for any cylinder set U there exist $\alpha_k\in\mathbb{R}$ and $Y(k)$ finite subset of $\mathbb{Z}^d$ such that $\mu(U)=\sum\alpha_k\hat{\mu}(Y(k))$, which implies the last statement of Theorem 2.

4.3.1 Proof of Corollary 1

When $\lambda(\emptyset)=0$, Theorem 2 and Lemma 2 together with the fact that $\pi(\emptyset)=0$ imply that for any cylinder U that does not contain the point ${}^\infty 0^\infty$ we have

$$\mu(U)=\sum\alpha_i\left(\sum_{k=0}^{\infty}\mathbb{P}_{Y_0=Y(i)}\{Y_k=\emptyset|Y_{k-1}\neq\emptyset\}\right)=0.$$

Indeed, from the proof of Theorem 2 we get that $\pi(\emptyset)=0$ when $p(I_r)=0$. Also, in the case $p(I_r)>0$ we have that $\pi(\emptyset)=0$ since $\pi(\emptyset)=\frac{\lambda(\emptyset)}{p(I_r)}$. Finally note that $\mu({}^\infty 0^\infty)=1-\mu(\{0,1\}^{\mathbb{Z}^d}-{}^\infty 0^\infty)=1$ which finishes the proof.

4.4 Proof of Proposition 1

Since the $\{\pi(J)|J\subseteq I_r\}$ represent the transition probabilities of the dual process for all $J\in I_r$ one has $\pi(J)\ge 0$ and Proposition 1 is a simple consequence of the following Lemma.

Lemma 3 *The transition probabilities $\pi(.)$ of the dual process satisfy*

$$\pi(\emptyset)=\frac{p(\emptyset)}{\mathbf{D}}$$

$$\pi(i)=\frac{p(i)-p(\emptyset)}{\mathbf{D}}$$

$$\pi(i,j) = \frac{1}{\mathbf{D}}[p(i,j) + p(\emptyset) - p(i) - p(j)]$$

$$\pi(i,j,k) = \frac{1}{\mathbf{D}}[p(i,j,k) - p(\emptyset) + p(i) + p(j) + p(k) - p(i,j) - p(i,k) - p(j,k)]$$

$$\pi(i,j,k,l) = \frac{1}{\mathbf{D}}\Bigg[p(i,j,k,l) + p(\emptyset) - \sum_{l_0\in\{i,j,k,l\}} p(l_0) + \sum_{\{l_0,l_1\}\subset\{i,j,k,l\}} p(l_0,l_1) - \sum_{\{l_0,l_1,l_2\}\subset\{i,j,k,l\}} p(l_0,l_1,l_2)\Bigg].$$

More generally, for any $0 \le k \le |I_r| - 1$ *and for any* $j_0, \ldots, j_k \in I_r$

$$\pi(j_0,\ldots,j_k) = \frac{1}{\mathbf{D}}\left[(-1)^{k+1}p(\emptyset) + \sum_{j=0}^{k}(-1)^{k-j}\sum_{\{l_0,\ldots,l_j\}\subset\{j_0,\ldots,j_k\}} p(l_0,\ldots,l_j)\right].$$

Proof of Lemma 3 From the proof of Theorem 2 a PCA belongs to class $\mathfrak{C}$ if and only if the transitions probabilities $p(.)$ and $\pi(.)$ satisfy the set of equations (1.8). We use mathematical induction to solve the set of equations (1.8). For the two first iterations it is easily seen that $\pi(\emptyset) = \frac{p(\emptyset)}{\mathbf{D}}$, $\pi(i) = \frac{p(i)-p(0)}{\mathbf{D}}$ and $\pi(i,j) = \frac{1}{\mathbf{D}}[p(i,j) + p(0) - p(i) - p(j)]$. Then suppose that the order k is true:

$$\pi(j_0,\ldots,j_k) = \frac{1}{\mathbf{D}}\left[(-1)^{k+1}p(\emptyset) + \sum_{j=0}^{k}(-1)^{k-j}\sum_{(l_0,\ldots,l_j)\in\{j_0,\ldots,j_k\}} p(l_0,\ldots,l_j)\right].$$

Using (1.8) we obtain that $\pi(j_0,\ldots,j_{k+1})$ equals

$$\frac{1}{\mathbf{D}}\left[p(j_0,\ldots,j_{k+1}) - d\pi(\emptyset) - \mathbf{D}\sum_{j=0}^{k}\left(\sum_{(l_0,\ldots,l_j)\in\{j_0,\ldots,j_{k+1}\}}\pi(l_0,\ldots,l_j)\right)\right]. \tag{1.9}$$

Then we suppose the rank k true and use (1.9) to obtain that the term in $p(\emptyset)$ in $\pi(j_0,\ldots,j_{k+1})$ is

$$\begin{aligned}
&-p(\emptyset) - \sum_{i=0}^{k}\left(\sum_{l_0,\ldots,l_i\in\{j_0,\ldots,j_{k+1}\}}(-1)^{i+1}p(\emptyset)\right)\\
&= p(\emptyset)\left(-1 - \sum_{i=0}^{k} C_{i+1}^{k+2}(-1)^{i+1}\right)\\
&= p(\emptyset)\Big(-1 + C_0^{k+2}(-1)^0 + C_{k+2}^{k+2}(-1)^{k+2} - (1-1)^{k+2}\Big)\\
&= (-1)^{k+2}p(\emptyset),
\end{aligned}$$

where the constants C_i^k represent the binomial coefficients. Next we obtain that the term in $\sum_{l_0\in\{j_0,\ldots,j_{k+1}\}} p(l_0)$ in $\pi(j_0,\ldots,j_{k+1})$ is equal to

$$\begin{aligned}
&-\sum_{i=0}^{k}\sum_{(l_0,\ldots,l_i)\in\{j_0,\ldots,j_{k+1}\}}\left(\sum_{h_0\in\{l_0,\ldots,l_i\}} p(h_0)\right)(-1)^i\\
&=-\sum_{l_0\in\{j_0,\ldots,j_{k+1}\}} p(l_0)\left(\sum_{i=0}^{k} C_i^{k+1}(-1)^i\right)\\
&=-\sum_{l_0\in\{j_0,\ldots,j_{k+1}\}} p(l_0)\left((1-1)^{k+1}-C_{k+1}^{k+1}(-1)^{k+1}\right)\\
&=\sum_{l_0\in\{j_0,\ldots,j_{k+1}\}} p(l_0)(-1)^{k+1}.
\end{aligned}$$

Note that C_i^{k+1} represents the number of ways to choose $l_1,\ldots,l_i$ in $j_1,\ldots,j_{k+1}$ when we have chosen l_0 and j_0. More generally, for $0\le M\le k$, the term in $\sum_{(l_0,\ldots,l_M)\in\{j_0,\ldots,j_{k+1}\}}$ $p(l_0,\ldots,l_M)$ in $\pi(j_0,\ldots,j_{k+1})$ is equal to

$$\begin{aligned}
&-\sum_{i=M}^{k}\sum_{(l_0,\ldots,l_i)\in\{j_0,\ldots,j_{k+1}\}}\left(\sum_{(h_0,\ldots,h_M)\in\{l_0,\ldots,l_j\}} p(h_0,\ldots,h_M)\right)(-1)^{i-M}\\
&=-\sum_{(l_0,\ldots,l_M)\in\{j_0,\ldots,j_{k+1}\}} p(l_0,\ldots,l_M)\left(\sum_{i=0}^{k-M} C_i^{k+1-M}(-1)^i\right)\\
&=-\sum_{(l_0,\ldots,l_M)\in\{j_0,\ldots,j_{k+1}\}} p(l_0,\ldots,l_M)\left((1-1)^{k+1-M}-(-1)^{k+1-M}\right)\\
&=\sum_{(l_0,\ldots,l_M)\in\{j_0,\ldots,j_{k+1}\}} p(l_0,\ldots,l_M)(-1)^{k+1-M},
\end{aligned}$$

which finishes the proof. □

5 Decay of Correlation

For the sake of simplicity we study the decay of correlation for PCA with state space $\{0,1\}^{\mathbb{Z}}$. An extension of this result to the multi-dimensional case is straightforward but requires too much notation.

5.1 Proof of Theorem 3

The proof of Theorem 3 requires the following two results. The second one is new and is a key point for the proof of Theorem 3. The first one seems to be well known. However, its proof can not be found or at least it is quite hard to be found so we provide a proof of that result.

Recall that μ stands for the unique invariant measure of an ergodic PCA.

Proposition 2 *Every invariant measure of an ergodic PCA is shift-invariant.*

Lemma 4 *Let* $[U]_0$ *and* $[V]_0$ *be two cylinders. If* $\mu([U]_0) = \sum \alpha_i \hat{\mu}(A_i), \mu([V]_0) = \sum \beta_i \hat{\mu}(B_i)$ *and* $t \geq |U| + |V|$, *then*

$$\mu([U]_0 \cap \sigma^{-t}[V]_0) = \mu([U]_0 \cap [V]_t) = \sum \alpha_i \hat{\mu}(A_i)(*, t) \sum \beta_i \hat{\mu}(B_i),$$

where

$$\sum \alpha_i \hat{\mu}(A_i)(*, t) \sum \beta_i \hat{\mu}(B_i) := \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}).$$

Proof of Theorem 3 If $\mathbf{D} = 0$, then $p(\emptyset) = 0$. From Corollary 1, $\mu = \delta_0$ and μ has exponential decay of correlation. For the remainder of this proof we therefore take $0 < \mathbf{D} = p(I_r) < 1$.

For any finite subset E of $\mathbb{Z}$ and $s \in \mathbb{Z}$, define $E + s := \{x + s : x \in E\}$. We claim that for any finite subsets E and F, if $t \geq 2Nr + |E| + |F|$ we have

$$\left|\hat{\mu}(E \cup \{F + t\}) - \hat{\mu}(E) \times \hat{\mu}(F)\right| \leq \mathbf{D}^{N+1} \frac{1}{1 - \mathbf{D}}.$$

The proof of this claim uses Theorems 1 and 2 which together say that for any finite subset $E \subset \mathbb{Z}$, $\hat{\mu}(E) = \mathbb{P}_{\eta_0 = E}\{\eta_\tau = \emptyset\}$. This, in turn, implies that

$$\hat{\mu}(E) = \sum_{k=0}^{\infty} \mathbb{P}_{\eta_0 = E}\{\tau = k\},$$

where τ is the hitting time for the process $\eta_\cdot$. In fact, by Lemma 1, for any integer $N > 0$ we have

$$\left|\hat{\mu}(E) - \sum_{k=0}^{N} \mathbb{P}_{\eta_0 = E}\{\tau = k\}\right| \leq \mathbf{D}^{N+1} \frac{1}{1 - \mathbf{D}}.$$

Note that if $s \geq 2ri + |E| + |F|$, where i is any positive integer, then

$$\mathbb{P}_{\eta_0 = E \cup \{F+s\}}\{\tau = i\} = \mathbb{P}_{\eta_0 = E}\{\tau = i\} \times \sum_{j=0}^{i} \mathbb{P}_{\eta_0 = \{F+s\}}\{\tau = j\}$$
$$+ \mathbb{P}_{\eta_0 = \{F+s\}}\{\tau = i\} \times \sum_{j=0}^{i} \mathbb{P}_{\eta_0 = E \cup \{F+s\}}\{\tau = j\}.$$

It follows that if $s \geq |E| + |F| + 2N \times r$, then

$$\sum_{i=0}^{N} \mathbb{P}_{E_0 = E \cup \{F+s\}}\{\tau = i\} = \sum_{i=0}^{N} \mathbb{P}_{E_0 = E}\{\tau = i\} \times \sum_{i=0}^{N} \mathbb{P}_{E_0 = \{F+s\}}\{\tau = i\}.$$

This easily implies

$$\left|\hat{\mu}(E \cup \{F + s\}) - \hat{\mu}(E) \times \hat{\mu}(F)\right| \leq \mathbf{D}^{N+1} \frac{1}{1 - \mathbf{D}}, \tag{1.10}$$

for $s \geq |E| + |F| + 2N \times r$, which proves our claim.

By Lemma 2, for any pair of cylinders $[U]_0$ and $[V]_0$, there exist finite sequences of sets (A_i) and (B_i) and finite sequences of real numbers α_i and β_i such that

$$\mu([U]_0) = \sum \alpha_i \hat{\mu}(A_i)$$

and

$$\mu([V]_0) = \sum \beta_i \hat{\mu}(B_i).$$

Thus, by inequality (1.10),

$$\Big|\alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + s\} - \alpha_i \hat{\mu}(A_i) \times \beta_j \hat{\mu}(B_j))\Big| \le |\alpha_i \beta_j| \mathbf{D}^{N+1} \frac{1}{1-\mathbf{D}}$$

for any pair of subsets A_i and B_j of $\mathbb{Z}$ and for any $s \ge |U| + |V| + 2Nr$.

It follows from this that

$$\Big|\sum_{i,j} \alpha_i \beta_j \hat{\mu}\big(A_i \cup \{B_i + s\}\big) - \sum_i \alpha_i \hat{\mu}(A_i) \times \sum_j \beta_j \hat{\mu}(B_j)\Big| \le F(U, V)\mathbf{D}^N,$$

where $F(U, V) = \sum_{i,j} |\alpha_i \beta_j| \frac{\mathbf{D}}{1-\mathbf{D}}$.

Using Lemma 4, if $t \ge |U| + |V|$ we obtain

$$\Big|\mu\left([U]_0 \cap \sigma^{-t}[V]_0\right) - \mu([U]_0) \times \mu([V]_0)\Big| \le K(U, V) \exp\Big(-t \times \frac{\ln(1/\mathbf{D})}{2r}\Big),$$

where $K(U, V) = F(U, V)\mathbf{D}^{-(\frac{|U|+|V|}{2r})}$.

Finally, it follows from Proposition 2 that the invariant measure is shift-invariant and that the exponential decay of correlations of cylinders implies the mixing property. □

5.1.1 Proof of Proposition 2

It is sufficient to show that for any cylinder $[U]_t$, where $U \in \{0, 1\}^l$ for some $l \in \mathbb{N}$, we have

$$\mu(\sigma^{-1}[U]_t) = \mu([U]_t).$$

Since μ is the invariant measure of an ergodic PCA $\eta_{\cdot}$, there exits a sequence $(\mu_i)_{i \in \mathbb{N}}$ which converges in the weak* topology to μ, where μ_i is the distribution of a PCA $\eta_{\cdot}$ at time i starting from an initial distribution μ_0. It follows that for any cylinder $[U]_t$ we have

$$\lim_{n \to \infty} \mu_n([U]_t) = \mu([U]_t).$$

Since for any positive integer i we have

$$\mu_i([U]_t) = \sum_{V_j \in \{0,1\}^{n+1+2ir}} \mu_0([V_j]_{t-ir}) \mathbb{P}_{\eta_0 \in [V_j]_{t-ir}} \{\eta_i \in [U]_t\},$$

we can choose μ_0 as a shift-invariant probability measure. Hence, for any positive integer i and any cylinder $[U]_t$ we have

$$\mu_i([U]_t) = \mu_i(\sigma^{-1}[U]_t),$$

which finishes the proof.

5.1.2 Proof of Lemma 4

We prove the lemma using the principle of mathematical induction. First we prove it for the case $|U| = 1$ and $|V| = 1$. Note that for any finite set $\{j_0, \ldots, j_k\}$, with $j_0, \ldots, j_k \in \mathbb{Z}$,

$$\begin{aligned}\hat{\mu}(\{j_0, \ldots, j_k\}) &= \int_{\{0,1\}^{\mathbb{Z}}} H(\{j_0, \ldots, j_k\}, x) d\mu(x) \\ &= \mu(\{\cap [1]_j \,|\, j \in \{j_0, \ldots, j_k\}\})\end{aligned}$$

and observe that for any $k \geq 1 \in \mathbb{N}$ we have

$$\mu([1]_0 \cap [1]_k) = \hat{\mu}(\{0\} \cup \{k\}).$$

Since $\mu([0]_k) = 1 - \hat{\mu}(\{k\}) = \hat{\mu}(\emptyset) - \hat{\mu}(\{k\})$ and $\mu([1]_0 \cap [0]_k) = \mu([1]_0) - \mu([1]_0 \cap [1]_k) = \hat{\mu}(\{0\}) - \hat{\mu}(\{0, k\})$ we get, again, that

$$\mu([1]_0 \cap [0]_k) = \hat{\mu}(\{0\} \cup \emptyset) - \hat{\mu}(\{0\} \cup \{k\}).$$

Furthermore, we have $\mu([0]_0 \cap [1]_k) = \hat{\mu}(\{1\}) - \hat{\mu}(\{0, k\})$.

Finally, note that

$$\begin{aligned}\mu([0]_0 \cap [0]_k) &= 1 - \mu([1]_0 \cap [1]_k) - \mu([1]_0 \cap [0]_k) - \mu([0]_0 \cap [1]_k) \\ &= \hat{\mu}(\emptyset) - \hat{\mu}(\{0\} \cup \{k\}) - \hat{\mu}(\{0\}) + \hat{\mu}(\{0\} \cup \{k\}) - \hat{\mu}(\{k\}) \\ &\quad + \hat{\mu}(\{0\} \cup \{k\}) \\ &= \hat{\mu}(\emptyset) + \hat{\mu}(\{0\} \cup \{k\}) - \hat{\mu}(\{0\}) - \hat{\mu}(\{k\}) \\ &= \left[\hat{\mu}(\emptyset) - \hat{\mu}(\{0\})\right] (*, t) \left[\hat{\mu}(\emptyset) - \hat{\mu}(\{k\})\right],\end{aligned}$$

which finishes the proof in the case $|U| = |V| = 1$.

Now, suppose that $\mu([U]_0 \cap \sigma^{-t}[V]_0) = \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\})$ is true for $|U| = |V| = n$. Consider $t \geq 2n + 1$ and let $[U]_0, [V]_0$ be two cylinders such that $\mu([U]_0) = \sum \alpha_i \hat{\mu}(A_i)$ and $\mu([V]_0) = \sum \beta_j \hat{\mu}(B_j)$. Since $\mu([U1]_0) = \sum \alpha_i \hat{\mu}(A_i \cup \{|U|\})$, we get

$$\mu([U1]_0 [V1]_t) = \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_j + t\} \cup \{|U|, |V| + t\}).$$

Noting that $\mu([V1]_t) = \sum \alpha_i \hat{\mu}(\{B_i + t\} \cup \{|V| + t\})$ we get the desired result for the case $[U1]_0 \cap [V1]_t$.

The result for the case $[U1]_0 \cap [V0]_t$ follows by noting that

$$\begin{aligned}\mu([U1]_0 \cap [V0]_t) &= \mu([U1]_0 \cap [V]_t) - \mu([U1]_0 \cap [V1]_t) \\ &= \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{|U|\} \cup \{B_j + t\}) \\ &\quad - \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_j + t\} \cup \{|U|, |V| + t\}) \\ &= \sum \alpha_i \hat{\mu}(A_i \cup \{|U|\}) (*, t) \sum \alpha_i \hat{\mu}(\{B_i + t\}) \\ &\quad - \sum \alpha_i \hat{\mu}(\{B_i + t\} \cup \{|V| + t\}).\end{aligned}$$

It can also be shown that

$$\mu([U0]_0 \cap [V1]_t) = \left[\sum \alpha_i \hat{\mu}(A_i) - \sum \alpha_i \hat{\mu}(A_i \cup \{|U|\})\right] \times (*, t)\left[\sum \alpha_i \hat{\mu}(\{B_i + t\} \cup \{|V| + t\})\right].$$

Finally, using that

$$\mu([U0]_0 \cap [V0]_t) = \mu([U]_0 \cap [V]_t) - \mu([U1]_0 \cap [V0]_t) - \mu([U0]_0 \cap [V1]_t) - \mu([U1]_0 \cap [V1]_t)$$

we can show that

$$\mu([U0]_0 \cap [V0]_t) = \left[\sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}) - \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\} \cup \{|U|\})\right](*, t) \times \left[\sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}) - \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}) \cup \{|V| + t\}\right]$$

which finishes the proof.

Final Questions

(i) Is there exist an ergodic PCA such that the unique invariant measure is not shift-mixing?
(ii) Is there exist an ergodic PCA such that the invariant measure has non-exponential decay of correlation?

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